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Buckling of a Rectangular Column on the Basis of Murnaghan's Formula for Strain Energy

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BUCKLING OF A RECTANGULAR COLUMN ON THE BASIS OF
MURNAGHAN'S FORMULA FOR STRAIN ENERGY

Samuel Lubkin

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TABLE OF CONTENTS

	Page
1. Introduction	1
2. Strain Energy Due to Perturbations	3
3. Conditions of Critical Strain	8
4. An Upper Limit of Stable Strain	12
5. Buckling Criteria Below Limit Strain	16
6. Evaluation of Buckling Strain for Small Sections	25
Footnotes	30

1. Introduction.

The classical theory of buckling of columns¹ is based upon the assumption of small deformations and cross-sections of dimensions so small as compared to the length that simple bending formulae are applicable. By analyzing the change in strain energy caused by small perturbations from the simple compressed state,² buckling criteria can be derived which are valid for large strains and cross-sections. The derivations are, however, generally based upon the classical formula³ for strain energy density. Prof. John⁴ admits a wide class of strain energy density relations which includes the classical (termed by him "standard") but does not include strain energy density functions of the type suggested by Murnaghan.⁵ When I first considered these latter density functions, I carelessly assumed that they fitted experimental results, particularly those obtained by Bridgman⁶ for high hydrostatic pressures, more closely than the classical density function. Re-examination, however, indicated that this was true only for the more complicated formulae in Murnaghan's article which involve more than two elastic constants. Murnaghan's simpler, two-constant, strain energy density function actually gives poorer agreement with the available experimental data than does the classical density function.

The present paper covers the evaluation of plane strain buckling criteria in a rectangular column assuming

the strain energy density formula derived by Murnaghan which is closest to the classical (involving two invariants of the strain matrix with, therefore, two elastic constants corresponding to the Lamé constants). Since, in a certain sense,⁷ the Murnaghan formula is simpler than the classical, it was hoped that analysis would prove easier and thus more difficult problems could be resolved by its use. This was found not to be the case. In fact, for the particular problem here considered, I could only obtain a series solution when working with Murnaghan's strain energy density whereas I was able to obtain an explicit solution for the equivalent problem using the classical density function. As expected, the Murnaghan formula leads to smaller values of buckling strains than does the classical and the difference is unimportant for small strains at buckling (small cross-section dimensions compared to length) for which both reduce to the well-known Euler condition for buckling of columns.

I am indebted to Dr. Sensenig for careful perusal of my original manuscript and constructive criticism on a number of points. I must also express my gratitude to Prof. Stoker for his unfailing encouragement throughout the study.

2. Strain Energy Due to Perturbations.

The Murnaghan strain energy density formula here used is:

$$\begin{aligned} 2\omega = & \lambda[(1+e_1)^2 + (1+e_2)^2 + (1+e_3)^2 - 3]^2 \\ & + 2\mu\{[(1+e_1)^2-1]^2 + [(1+e_2)^2-1]^2 + [(1+e_3)^2-1]^2\} , \end{aligned}$$

in which e_1 , e_2 , and e_3 are the strains in the principal directions at any point in the strained body and λ and μ are elastic constants (Lamé). This may be compared to the classical formula:

$$2\omega = \lambda(e_1 + e_2 + e_3)^2 + 2\mu(e_1^2 + e_2^2 + e_3^2) .$$

We consider a column of rectangular section with ends constrained to remain plane and parallel. It is further assumed that there is zero shear forces over the ends and no forces on two opposite faces. The other two faces are constrained so that the normal strain to these faces is zero ($e_3 = 0$) in the simple compression condition prior to buckling. It is further assumed that the only admissible perturbations are those for which $e_3 = 0$ thus reducing the problem to two dimensions. This is the same condition as assumed by Fritz John.⁸

We take Cartesian coordinates x, y with the x axis as the axis of the column and the y axis perpendicular to the free faces. x is zero at one end of the column and L (the length of the column) at the other. y is zero

at the center of the cross-section and $\pm h/2$ at the faces. Both x and y refer to the unstrained column. In the x, y plane, we take displacements from the unstrained position as u along the x axis and v along the y axis so that point x, y moves to $x+u, y+v$ after deformation.

Now⁹

$$[(1+e_1) + (1+e_2)]^2 = (2+u_x+v_y)^2 + (u_y-v_x)^2$$

and

$$[(1+e_1) - (1+e_2)]^2 = (u_x - v_y)^2 + (u_y+v_x)^2$$

Hence

$$(1+e_1)^2 + (1+e_2)^2 = \frac{1}{2}[(2+u_x+v_y)^2 + (u_y-v_x)^2 + (u_x-v_y)^2 + (u_y+v_x)^2]$$

$$[(1+e_1)^2 - (1+e_2)^2]^2 = [(2+u_x+v_y)^2 + (u_y-v_x)^2][(u_x-v_y)^2 + (u_y+v_x)^2]$$

Thus

$$8\omega = (\lambda + \mu)[(1+e_1)^2 + (1+e_2)^2 - 2]^2 + \mu[(1+e_1)^2 - (1+e_2)^2]^2$$

and

$$32\omega = (\lambda + \mu)[(2+u_x+v_y)^2 + (u_y-v_x)^2 + (u_x-v_y)^2 + (u_y+v_x)^2 - 4]^2 + 4\mu[(2+u_x+v_y)^2 + (u_y-v_x)^2][(u_x-v_y)^2 + (u_y+v_x)^2]$$

Let

$$u = -ex + F(x, y)$$

$$v = \varepsilon y + G(x, y)$$

where

$$F(0, y) = F(L, y) = 0.$$

Then e determines the geometric boundary conditions at the ends of the column. Differentiating:

$$u_x = -e + F_x$$

$$u_y = F_y$$

$$v_x = G_x$$

$$v_y = \varepsilon + G_y$$

and

$$\begin{aligned} 32\omega = & (\lambda+\mu)\{[(1-e)+(1+\varepsilon)+(F_x+G_y)]^2 + [(1-e)-(1+\varepsilon)+(F_x-G_y)]^2 \\ & + 2F_y^2 + 2G_x^2 - 4\}^2 \\ & + 4\mu\{[(1-e)+(1+\varepsilon)+(F_x+G_y)]^2 + (F_y-G_x)^2\} \\ & \cdot \{[(1-e)-(1+\varepsilon)+(F_x-G_y)]^2 + (F_y+G_x)^2\} \end{aligned}$$

or

$$\begin{aligned} 8\omega = & (\lambda+\mu)\{[(1-e)^2+(1+\varepsilon)^2-2]+2(1-e)F_x+2(1+\varepsilon)G_y \\ & +(F_x^2+G_y^2+F_y^2+G_x^2)\}^2 \\ & + \mu\{[(1-e)+(1+\varepsilon)]^2+2[(1-e)+(1+\varepsilon)](F_x+G_y)+(F_x^2+G_y^2+F_y^2+G_x^2) \\ & + 2(F_xG_y-F_yG_x)\} \\ & \cdot \{[(1-e)-(1+\varepsilon)]^2+2[(1-e)-(1+\varepsilon)](F_x-G_y)+ \\ & +(F_x^2+G_y^2+F_y^2+G_x^2)-2(F_xG_y-F_yG_x)\} \\ = & \{(\lambda+\mu)[(1-e)^2+(1+\varepsilon)^2-2]^2 + \mu[(1-e)^2-(1+\varepsilon)^2]^2\} \\ & + 4\{(1-e)[(\lambda+2\mu)(1-e)^2+\lambda(1+\varepsilon)^2-2(\lambda+\mu)]F_x \\ & + (1+\varepsilon)[\lambda(1-e)^2+(\lambda+2\mu)(1+\varepsilon)^2-2(\lambda+\mu)]G_y\} \end{aligned}$$

$$\begin{aligned}
 & + 2\{[\lambda(\lambda+2\mu)(1-e)^2 + \lambda(1+\epsilon)^2 - 2(\lambda+\mu)]F_x^2 \\
 & + [\lambda(1-e)^2 + \lambda(\lambda+2\mu)(1+\epsilon)^2 - 2(\lambda+\mu)]G_y^2 \\
 & + [(\lambda+2\mu)(1-e)^2 + (\lambda+2\mu)(1+\epsilon)^2 - 2(\lambda+\mu)](F_y^2 + G_x^2) \\
 & + 4\lambda(1-e)(1+\epsilon)F_xG_y + 4\mu(1-e)(1+\epsilon)F_yG_x\} \\
 & + 4\{(\lambda+2\mu)[(1-e)F_x + (1+\epsilon)G_y](F_x^2 + G_y^2 + F_y^2 + G_x^2) \\
 & - 2\mu[(1+\epsilon)F_x + (1-e)G_y](F_xG_y - F_yG_x)\} \\
 & + \{(\lambda+2\mu)(F_x^2 + G_y^2 + F_y^2 + G_x^2)^2 - 4\mu(F_xG_y - F_yG_x)^2\} .
 \end{aligned}$$

For simple compression ($F = G = 0$), minimum energy requires:

$$\frac{\partial \omega}{\partial \epsilon} = 0$$

or

$$(\lambda+\mu)[(1-e)^2 + (1+\epsilon)^2 - 2] = \mu[(1-e)^2 - (1+\epsilon)^2]$$

or

$$(1+\epsilon)^2 = 1 + \frac{\lambda[1-(1-e)^2]}{\lambda + 2\mu} .$$

That simple compression, with this value of ϵ , satisfies both equilibrium and boundary conditions is easy to show. The x, y, and z axes are, obviously, principal directions of strain. Therefore, there are no shear stresses on the ends or faces of the column, nor on any internal parallelpiped parallel to the axes. The stresses in the directions of the x and y axes are given by $\frac{\partial \omega}{\partial e}$ and $\frac{\partial \omega}{\partial \epsilon}$ respectively and are, therefore, constant throughout the

column so that the latter is internally in equilibrium. Finally, the above value of ϵ corresponds to zero $\frac{\partial W}{\partial \epsilon}$ so that there are no normal stresses on the free faces of the column.

The above value of ϵ makes the coefficient of G_y vanish. Also, since $F = 0$ at $x = 0$ and $x = L$:

$$\int_0^L F_x \, dx = 0 .$$

The critical condition is one for which there exist small perturbations which reduce the total strain energy. It therefore suffices to consider the lowest order terms, which are not identically zero, in such perturbations. These are the quadratic terms. We thus need only consider the change in energy given by

$$\Delta W = \int_{-h/2}^{h/2} \int_0^L Q \, dx \, dy$$

and, substituting the value for $(1+\epsilon)^2$,

$$Q = \frac{[(\lambda^2 + 6\lambda\mu + 6\mu^2)(1-e)^2 - 2\mu(\lambda + \mu)]}{\lambda + 2\mu} F_x^2 + [2(\lambda + \mu) - \lambda(1-e)^2] G_y^2 \\ + \mu(1-e)^2 (F_y^2 + G_x^2) + 2(1-e)(1+\epsilon)(\lambda F_x G_y + \mu F_y G_x) .$$

3. Conditions at Critical Strain.

The column will buckle when there exist functions F and G which satisfy $F(0,y) = F(L,y) = 0$ but make ΔW negative. The critical strain is such a value of e that, for all lower values of e , the column is stable and, for all higher values, the column buckles.

If, in the expression for ΔW , we change variables to

$$\xi = \frac{x}{L} ,$$

$$\eta = \frac{y}{h} ,$$

we obtain

$$\Delta W = \int_{-1/2}^{1/2} \int_0^1 \bar{Q} \, d\xi \, d\eta ,$$

where

$$\begin{aligned} \bar{Q} = & \frac{[(\lambda^2 + 6\lambda\mu + 6\mu^2)(1-e)^2 - 2\mu(\lambda + \mu)]}{\lambda + 2\mu} \frac{h}{L} F_{\xi}^2 \\ & + [2(\lambda + \mu) - \lambda(1-e)^2] \frac{L}{h} G_{\eta}^2 + \mu(1-e)^2 \left(\frac{L}{h} F_{\eta}^2 + \frac{h}{L} G_{\xi}^2 \right) \\ & + 2(1-e)(1+e)(\lambda F_{\xi} G_{\eta} + \mu F_{\eta} G_{\xi}) . \end{aligned}$$

The only way in which the column dimensions enter in ΔW is, therefore, in the form of the ratio of h to L . Columns for which this ratio is the same thus have the same critical strain e .

As for the corresponding analysis using the standard energy formula, we now change variables to

$$\Theta = \frac{\pi}{L} x \quad \text{with range } 0 \text{ to } \pi$$

$$\phi = \frac{\pi}{L} y \quad \text{with range } -\frac{H}{2} \text{ to } \frac{H}{2}$$

where

$$H = \frac{\pi h}{L}$$

and take

$$F = \sum F_n(\phi) \sin n\Theta$$

$$G = G_0(\phi) + \sum G_n(\phi) \cos n\Theta$$

which are consistent with the boundary conditions at the ends of the column. The orthogonality properties of the trigonometric terms then permit us to write:

$$\int_0^L Q \, dx = \frac{\pi^2}{2L} [R_0(\phi) + \sum R_n(\phi)]$$

where

$$\begin{aligned} R_0 &= 2G_0'^2 [2(\lambda+\mu) - (1-e)^2] \\ R_n &= n^2 F_n^2 \frac{[(\lambda^2 + 6\lambda\mu + 6\mu^2)(1-e)^2 - 2\mu(\lambda+\mu)]}{\lambda+2\mu} \\ &\quad + G_n'^2 [2(\lambda+\mu) - \lambda(1-e)^2] + \mu(1-e)^2 (F_n'^2 + n^2 G_n^2) \\ &\quad + 2n(1-e)(1+\epsilon)(\lambda F_n G_n' - \mu F_n' G_n) \end{aligned}$$

in which primes denote differentiation with respect to ϕ .

Each R_n contributes to ΔW independently, having no common terms with any other. We are interested in values of e for which we can have

$$\Delta W_n = \frac{\pi}{2} \int_{-H/2}^{H/2} R_n \, d\phi < 0.$$

Obviously, ΔW_0 can never be negative. Furthermore, if we change the variable in ΔW_n to

$$\bar{\phi} = n\phi$$

we obtain

$$\Delta W_n = \frac{n^2\pi}{2} \int_{-\bar{H}/2}^{\bar{H}/2} \bar{R}_n d\bar{\phi}$$

in which

$$\bar{H} = nH$$

and $\bar{R}_n$ is identical with R , except for the subscripts on F and G . The inequality

$$\Delta W_n < 0$$

for a column of parameter H therefore leads to the same value of e as does the inequality

$$\Delta W_1 < 0$$

for a column of parameter nH , i.e. a column whose h/L is n times as great. We may thus restrict further analysis to the inequality

$$\Delta W_1 = \frac{\pi}{2} \int_{-H/2}^{H/2} R_1(F_1, G_1, F_1', G_1', e) d\phi < 0$$

and, from this point on, drop the subscript "1" from the quantities involved. We also ignore the positive factor $\pi/2$.

ΔW is minimized for

$$\frac{\partial R}{\partial F} = \left(\frac{\partial R}{\partial F'} \right)'$$

and

$$\frac{\partial R}{\partial G} = \left(\frac{\partial R}{\partial G'} \right)' .$$

That is

$$\frac{[(\lambda^2 + 6\lambda\mu + 6\mu^2)(1-e)^2 - 2\mu(\lambda + \mu)]}{\lambda + 2\mu} F + \lambda(1-e)(1+\epsilon)G'$$

$$= \mu(1-e)^2 F'' - \mu(1-e)(1+\epsilon)G'$$

and

$$\begin{aligned} & \mu(1-e)^2 G - \mu(1-e)(1+\epsilon)F' \\ &= [2(\lambda + \mu) - \lambda(1-e)^2]G'' + \lambda(1-e)(1+\epsilon)F' \end{aligned}$$

or, eliminating and substituting the value of $(1+\epsilon)^2$:

$$\begin{aligned} & (\lambda + 2\mu)(1-e)^2 [2(\lambda + \mu) - \lambda(1-e)^2]G^{iv} \\ &+ 2[2(\lambda + \mu)^2 - 5(\lambda + \mu)^2(1-e)^2 + (2\lambda^2 + 2\lambda\mu - \mu^2)(1-e)^4]G'' \\ &+ (1-e)^2 [(\lambda^2 + 6\lambda\mu + 6\mu^2)(1-e)^2 - 2\mu(\lambda + \mu)]G = 0. \end{aligned}$$

Let

$$\delta = e - \frac{e^2}{2}$$

then

$$(1-e)^2 = 1 - 2\delta$$

and

$$(1+\epsilon)^2 = \frac{(\lambda + 2\mu) + 2\lambda\delta}{\lambda + 2\mu}$$

making

$$aG^{iv} - 2bG'' + cG = 0$$

where

$$a = (\lambda + 2\mu)(1 - 2\delta)[(\lambda + 2\mu) + 2\lambda\delta] > 0$$

$$b = (\lambda + 2\mu)^2 - 2\delta(\lambda^2 + 6\lambda\mu + 7\mu^2) - 4\delta^2(2\lambda^2 + 2\lambda\mu - \mu^2)$$

$$c = (1 - 2\delta)[(\lambda + 2\mu)^2 - 2\delta(\lambda^2 + 6\lambda\mu + 6\mu^2)]$$

so that

$$ac - b^2 = 4\delta^2(\lambda + \mu)^3[(3\lambda + 7\mu) - 4\delta(\lambda + 5\mu) - 4\delta^2(5\lambda + \mu)]$$

4. An Upper Limit of Stable Strain.

If

$$\begin{aligned}\delta &> \frac{\sqrt{(\lambda+5\mu)^2 + (3\lambda+7\mu)(5\lambda+\mu)} - (\lambda+5\mu)}{2(5\lambda+\mu)} \\ &= \frac{4\sqrt{(\lambda+\mu)(\lambda+2\mu)} - (\lambda+5\mu)}{2(5\lambda+\mu)}\end{aligned}$$

then

$$b^2 - ac > 0 .$$

Now

$$\frac{db}{d\delta} = - 2[(\lambda^2 + 6\lambda\mu + 7\mu^2) + 4\delta(2\lambda^2 + 2\lambda\mu - \mu^2)]$$

$$< 0 \quad \text{for all admissible } \delta \quad (0 < \delta < 1/2).$$

Hence, for δ such that $b^2 - ac > 0$,

$$\begin{aligned}b(5\lambda+\mu)^2 &< (\lambda+2\mu)^2(5\lambda+\mu)^2 - (5\lambda+\mu)(\lambda^2+6\lambda\mu+7\mu^2) \\ &\quad \cdot [4\sqrt{(\lambda+\mu)(\lambda+2\mu)} - (\lambda+5\mu)] \\ &= - (2\lambda^2+2\lambda\mu-\mu^2)[4\sqrt{(\lambda+\mu)(\lambda+2\mu)} - (\lambda+5\mu)]^2 \\ &= - 4(\lambda+\mu)[(\lambda+2\mu)(\lambda^2-7\lambda\mu-12\mu^2) \\ &\quad + (\lambda^2+6\lambda\mu+17\mu^2)\sqrt{(\lambda+\mu)(\lambda+2\mu)}] \\ &\leq - 4(\lambda+\mu)\sqrt{\lambda+2\mu}[(\lambda^2+6\lambda\mu+17\mu^2)\sqrt{\lambda+\mu} \\ &\quad - |\lambda^2-7\lambda\mu-12\mu^2|\sqrt{\lambda+2\mu}] \end{aligned}$$

But

$$(\lambda+\mu)(\lambda^2+6\lambda\mu+17\mu^2)^2 = \lambda^5 + 13\lambda^4\mu + 82\lambda^3\mu^2 + 274\lambda^2\mu^3$$

$$\begin{aligned}\text{and} \quad &+ 493\lambda\mu^4 + 289\mu^5, \\ (\lambda+2\mu)(\lambda^2-7\lambda\mu-12\mu^2)^2 &= \lambda^5 - 12\lambda^4\mu - 3\lambda^3\mu^2 + 218\lambda^2\mu^3 + 480\lambda\mu^4 + 288\mu^5 \\ \text{so that} \quad &\end{aligned}$$

$$b < 0 .$$

Hence, one pair of roots of

$$ax^4 - 2bx^2 + c = 0$$

is

$$x = \pm i \alpha$$

where

$$\alpha^2 = \frac{-b + \sqrt{b^2 - ac}}{a} > 0$$

and an admissible value for G is:

$$G = A \cos \alpha \phi + B \sin \alpha \phi$$

making

$$G' = -A \alpha \sin \alpha \phi + B \alpha \cos \alpha \phi$$

$$G'' = -A \alpha^2 \cos \alpha \phi - B \alpha^2 \sin \alpha \phi .$$

But

$$(\lambda + \mu)(1-e)(1+\epsilon)F' = \mu(1-e)^2G - [2(\lambda + \mu) - \lambda(1-e)^2]G'' .$$

Thus

$$F' = I(A \cos \alpha \phi + B \sin \alpha \phi)$$

and

$$F = \frac{I}{\alpha} (A \sin \alpha \phi - B \cos \alpha \phi)$$

where

$$I = \frac{\mu(1-e)^2 + \alpha^2 [2(\lambda + \mu) - \lambda(1-e)^2]}{(\lambda + \mu)(1-e)(1+\epsilon)} .$$

Moreover,¹⁰

$$\int_{-H/2}^{H/2} R \, d\phi = \{ \mu(1-e)^2 FF' + [2(\lambda + \mu) - \lambda(1-e)^2] GG' + (\lambda - \mu)(1-e)(1+\epsilon)FG \} \Big|_{-H/2}^{H/2}$$

so that

$$\Delta W = L(A^2 - B^2) \sin \alpha H$$

where

$$\begin{aligned} L &= \mu(1-e)^2 \frac{I^2}{\alpha} - [2(\lambda+\mu) - \lambda(1-e)^2] \alpha + (\lambda-\mu)(1-e)(1+\epsilon) \frac{I}{\alpha} \\ &= \frac{\mu}{\alpha(\lambda+\mu)^2 [(\lambda+2\mu)+2\lambda\delta]} \{ (1-2\delta)[\lambda^2(\lambda+2\mu)+2\delta(\lambda^3-2\lambda\mu^2-2\mu^3)] \\ &\quad - 2\alpha^2 [(\lambda+2\mu)+2\lambda\delta][\lambda(\lambda+2\mu)+2\delta(\lambda^2+2\lambda\mu+2\mu^2)] \\ &\quad + \alpha^4(\lambda+2\mu)[(\lambda+2\mu)+2\lambda\delta]^2 \} . \end{aligned}$$

But

$$\alpha^4 = - \frac{2b\alpha^2 + c}{a} .$$

Hence

$$L = \frac{-2\mu}{\alpha(\lambda+\mu)} [(\lambda+2\mu) - \delta(\lambda+5\mu) - 2\delta^2(3\lambda+\mu)] \left\{ \frac{\mu}{(\lambda+2\mu)+2\lambda\delta} + \frac{\alpha^2}{1-2\delta} \right\} .$$

Thus $L \neq 0$ except for the single positive value of δ which satisfies

$$(\lambda+2\mu) - \delta(\lambda+5\mu) - 2\delta^2(3\lambda+\mu) = 0 .$$

Also $\sin \alpha H \neq 0$ except for the particular, isolated values of δ which make αH a multiple of π . Thus $L \sin \alpha H \neq 0$ except for special, discrete values of δ . If $L \sin \alpha H < 0$, then ΔW becomes negative by choosing $B = 0$, $A \neq 0$. If $L \sin \alpha H > 0$, then ΔW becomes negative by choosing $A = 0$, $B \neq 0$. Hence, for any H , no matter how large, buckling occurs for

$$\delta > \frac{4 \sqrt{(\lambda+\mu)(\lambda+2\mu)} - (\lambda+5\mu)}{2(5\lambda+\mu)}$$

$$= \frac{3}{10} \quad \text{for } \lambda \gg \mu$$

$$= \frac{1}{3} \sqrt{6} - \frac{1}{2} = .316 \quad \text{for } \lambda = \mu$$

$$= 252 - \frac{5}{2} = .328 \quad \text{for } \lambda \ll \mu .$$

We will find, however, that the actual buckling limit is still lower, corresponding to the condition $ac > b^2$, treated in the next section.

5. Buckling Criteria below Limit Strain.

If

$$\delta < \frac{4\sqrt{(\lambda+\mu)(\lambda+2\mu)} - (\lambda+5\mu)}{2(5\lambda+\mu)}$$

then

$$ac - b^2 > 0$$

so that

$$ac > b^2 > 0$$

and

$$\frac{c}{a} > \frac{b^2}{a^2} > 0.$$

Thus

$$\sqrt{\frac{c}{a}} > \left| \frac{b}{a} \right|.$$

The roots of $ax^4 - 2bx^2 + c = 0$ are then

$$x = \pm \sqrt{\frac{1}{2} \left(\sqrt{\frac{c}{a}} + \frac{b}{a} \right)} \pm i \sqrt{\frac{1}{2} \left(\sqrt{\frac{c}{a}} - \frac{b}{a} \right)}$$

where all sign combinations are valid. Let

$$\alpha = \sqrt{\frac{1}{2} \left(\sqrt{\frac{c}{a}} + \frac{b}{a} \right)}$$

$$\longrightarrow 1 - \frac{(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2} \delta \quad \text{for } \delta \ll 1$$

$$\longrightarrow 0 \quad \text{for } ac - b^2 \longrightarrow 0$$

(since $b < 0$ at this limit)

and

$$\beta = \sqrt{\frac{1}{2} \left(\sqrt{\frac{c}{a}} - \frac{b}{a} \right)}$$

$$\longrightarrow \frac{(\lambda+\mu)\delta}{(\lambda+2\mu)^2} \sqrt{(\lambda+\mu)(3\lambda+7\mu)} \quad \text{for } \delta \ll 1$$

$$\longrightarrow \sqrt{-\frac{b}{a}} \quad \text{for } ac - b^2 \longrightarrow 0.$$

Then

$$\begin{aligned}
 G &= A \cosh \alpha \phi \cos \beta \phi + B \sinh \alpha \phi \sin \beta \phi \\
 &\quad + C \sinh \alpha \phi \cos \beta \phi + D \cosh \alpha \phi \sin \beta \phi \\
 G' &= (\alpha A + \beta B) \sinh \alpha \phi \cos \beta \phi + (\alpha B - \beta A) \cosh \alpha \phi \sin \beta \phi \\
 &\quad + (\alpha C + \beta D) \cosh \alpha \phi \cos \beta \phi + (\alpha D - \beta C) \sinh \alpha \phi \sin \beta \phi \\
 G'' &= [(\alpha^2 - \beta^2)A + 2\alpha\beta B] \cosh \alpha \phi \cos \beta \phi \\
 &\quad + [(\alpha^2 - \beta^2)B - 2\alpha\beta A] \sinh \alpha \phi \sin \beta \phi \\
 &\quad + [(\alpha^2 - \beta^2)C + 2\alpha\beta D] \sinh \alpha \phi \cos \beta \phi \\
 &\quad + [(\alpha^2 - \beta^2)D - 2\alpha\beta C] \cosh \alpha \phi \sin \beta \phi .
 \end{aligned}$$

Let

$$\begin{aligned}
 I &= \frac{[2(\lambda + \mu) - \lambda(1-e)^2](\alpha^2 - \beta^2) - \mu(1-e)^2}{(\lambda + \mu)(1-e)(1+\epsilon)} \\
 &= \frac{b - \mu(\lambda + 2\mu)(1-2\delta)^2}{(\lambda + \mu)(1-2\delta)\sqrt{a}}
 \end{aligned}$$

and

$$\begin{aligned}
 J &= \frac{2\alpha\beta[2(\lambda + \mu) - \lambda(1-e)^2]}{(\lambda + \mu)(1-e)(1+\epsilon)} \\
 &= \frac{\sqrt{ac-b^2}}{(\lambda + \mu)(1-2\delta)\sqrt{a}}
 \end{aligned}$$

Then $I^2 + J^2 = 1$, and

$$\begin{aligned}
 F' &= - (IA + JB) \cosh \alpha \phi \cos \beta \phi + (JA - IB) \sinh \alpha \phi \sin \beta \phi \\
 &\quad - (IC + JD) \sinh \alpha \phi \cos \beta \phi + (JC - ID) \cosh \alpha \phi \sin \beta \phi ;
 \end{aligned}$$

$$\begin{aligned}
 F = & -\sqrt{\frac{a}{c}} \{ [\alpha I + \beta J]A + (\alpha J - \beta I)B \} \sinh \alpha \phi \cos \beta \phi \\
 & + [(\alpha I + \beta J)B - (\alpha J - \beta I)A] \cosh \alpha \phi \sin \beta \phi \\
 & + [(\alpha I + \beta J)C + (\alpha J - \beta I)D] \cosh \alpha \phi \cos \beta \phi \\
 & + [(\alpha I + \beta J)D - (\alpha J - \beta I)C] \sinh \alpha \phi \sin \beta \phi \} .
 \end{aligned}$$

Now

$$\begin{aligned}
 \sqrt{\frac{c}{a}} FF' \Big|_{-H/2}^{H/2} = & \{ (IA + JB) [(\alpha I + \beta J)A + (\alpha J - \beta I)B] \\
 & + (IC + JD) [(\alpha I + \beta J)C + (\alpha J - \beta I)D] \} \sinh \alpha H \cos^2 \frac{\beta H}{2} \\
 & - \{ (JA - IB) [(\alpha I + \beta J)B - (\alpha J - \beta I)A] \\
 & - (JC - ID) [(\alpha I + \beta J)D - (\alpha J - \beta I)C] \} \sinh \alpha H \sin^2 \frac{\beta H}{2} \\
 & + \{ (IA + JB) [(\alpha I + \beta J)B - (\alpha J - \beta I)A] \\
 & - (JC - ID) [(\alpha I + \beta J)C + (\alpha J - \beta I)D] \} \cosh^2 \frac{\alpha H}{2} \sin \beta H \\
 & + \{ -(JA - IB) [(\alpha I + \beta J)A + (\alpha J - \beta I)B] \\
 & + (IC + JD) [(\alpha I + \beta J)D - (\alpha J - \beta I)C] \} \sinh^2 \frac{\alpha H}{2} \sin \beta H \\
 = & \frac{\alpha}{2} (A^2 + B^2 + C^2 + D^2) \sinh \alpha H \\
 & + \frac{1}{2} [\alpha (I^2 - J^2) + 2\beta IJ] (A^2 - B^2 + C^2 - D^2) \sinh \alpha H \cos \beta H \\
 & - [\beta (I^2 - J^2) - 2\alpha IJ] (AB + CD) \sinh \alpha H \cos \beta H \\
 & + \frac{\beta}{2} (A^2 + B^2 - C^2 - D^2) \sin \beta H \\
 & + \frac{1}{2} [\beta (I^2 - J^2) - 2\alpha IJ] (A^2 - B^2 + C^2 - D^2) \cosh \alpha H \sin \beta H \\
 & + [\alpha (I^2 - J^2) + 2\beta IJ] (AB + CD) \cosh \alpha H \sin \beta H .
 \end{aligned}$$

Also

$$\begin{aligned}
 GG' \Big|_{-H/2}^{H/2} &= \{A(\alpha A + \beta B) + C(\alpha C + \beta D)\} \sinh \alpha H \cos^2 \frac{\beta H}{2} \\
 &\quad + \{B(\alpha B - \beta A) + D(\alpha D - \beta C)\} \sinh \alpha H \sin^2 \frac{\beta H}{2} \\
 &\quad + \{B(\alpha A + \beta B) + C(\alpha D - \beta C)\} \sinh^2 \frac{\alpha H}{2} \sin \beta H \\
 &\quad + \{A(\alpha B - \beta A) + D(\alpha C + \beta D)\} \cosh^2 \frac{\alpha H}{2} \sin \beta H \\
 &= \frac{\alpha}{2} (A^2 + B^2 + C^2 + D^2) \sinh \alpha H \\
 &\quad + \frac{\alpha}{2} (A^2 - B^2 + C^2 - D^2) \sinh \alpha H \sin \beta H \\
 &\quad + \beta (AB + CD) \sinh \alpha H \sin \beta H \\
 &\quad - \frac{\beta}{2} (A^2 + B^2 - C^2 - D^2) \sin \beta H \\
 &\quad - \frac{\beta}{2} (A^2 - B^2 + C^2 - D^2) \cosh \alpha H \sin \beta H \\
 &\quad + \alpha (AB + CD) \cosh \alpha H \sin \beta H
 \end{aligned}$$

and

$$\begin{aligned}
 \sqrt{\frac{c}{a}} FG \Big|_{-H/2}^{H/2} &= \{-A[(\alpha I + \beta J)A + (\alpha J - \beta I)B] \\
 &\quad - C[(\alpha I + \beta J)C + (\alpha J - \beta I)D]\} \sinh \alpha H \cos^2 \frac{\beta H}{2} \\
 &\quad + \{-B[(\alpha I + \beta J)B - (\alpha J - \beta I)A] \\
 &\quad - D[(\alpha I + \beta J)D - (\alpha J - \beta I)C]\} \sinh \alpha H \sin^2 \frac{\beta H}{2} \\
 &\quad + \{-A[(\alpha I + \beta J)B - (\alpha J - \beta I)A] \\
 &\quad - D[(\alpha I - \beta J)C + (\alpha J - \beta I)D]\} \cosh^2 \frac{\alpha H}{2} \sin \beta H \\
 &\quad + \{-B[(\alpha I + \beta J)A + (\alpha J - \beta I)B]
 \end{aligned}$$

$$\begin{aligned}
 &= -C[(\alpha I + \beta J)D - (\alpha J - \beta I)C] \sinh^2 \frac{\alpha H}{2} \sin \beta H \\
 &= -\frac{1}{2} (\alpha I + \beta J) (A^2 + B^2 + C^2 + D^2) \sinh \alpha H \\
 &\quad - \frac{1}{2} (\alpha I + \beta J) (A^2 - B^2 + C^2 - D^2) \sinh \alpha H \cos \beta H \\
 &\quad - (\alpha J - \beta I) (AB + CD) \sinh \alpha H \cos \beta H \\
 &\quad + \frac{1}{2} (\alpha J - \beta I) (A^2 + B^2 - C^2 - D^2) \sin \beta H \\
 &\quad + \frac{1}{2} (\alpha J - \beta I) (A^2 - B^2 + C^2 - D^2) \cosh \alpha H \sin \beta H \\
 &\quad - (\alpha I + \beta J) (AB + CD) \cosh \alpha H \sin \beta H
 \end{aligned}$$

Making

$$\begin{aligned}
 2\sqrt{\frac{c}{a}} \Delta W &= S(A^2 + B^2 + C^2 + D^2) \sinh \alpha H \\
 &\quad - T(A^2 + B^2 - C^2 - D^2) \sin \beta H \\
 &\quad + (S-U)[(A^2 - B^2 + C^2 - D^2) \sinh \alpha H \cos \beta H \\
 &\quad \quad + 2(AB + CD) \cosh \alpha H \sin \beta H] \\
 &\quad - (T+V)[(A^2 - B^2 + C^2 - D^2) \cosh \alpha H \sin \beta H \\
 &\quad \quad - 2(AB + CD) \sinh \alpha H \cos \beta H]
 \end{aligned}$$

where

$$\begin{aligned}
 S &= \alpha \mu (1-e)^2 + \alpha \sqrt{\frac{c}{a}} [2(\lambda + \mu) - \mu(1-e)^2] - (\lambda - \mu)(1-e)(1+\epsilon)(\alpha I + \beta J) \\
 T &= -\beta \mu (1-e)^2 + \beta \sqrt{\frac{c}{a}} [2(\lambda + \mu) - \lambda(1-e)^2] - (\lambda - \mu)(1-e)(1+\epsilon)(\alpha J - \beta I) \\
 U &= \mu(1-e)^2 [\alpha(1-I^2 + J^2) - 2\beta IJ] \\
 &= 2\mu(1-e)^2 J(\alpha J - \beta I)
 \end{aligned}$$

$$V = u(1-e)^2[\beta(1-I^2 + J^2) + 2\alpha IJ]$$

$$= 2u(1-e)^2 J(\alpha I + \beta J) .$$

Hence

$$2\sqrt{\frac{c}{a}} \Delta W = L(A^2 + C^2) + 2M(AB + CD) + N(B^2 + D^2) + 2P$$

where

$$L = (S-U) \sinh \alpha H \cos \beta H - (T+V) \cosh \alpha H \sin \beta H$$

$$+ S \sinh \alpha H - |T \sin \beta H|$$

$$M = (T+V) \sinh \alpha H \cos \beta H + (S-U) \cosh \alpha H \sin \beta H$$

$$N = (T+V) \cosh \alpha H \sin \beta H - (S-U) \sinh \alpha H \cos \beta H$$

$$+ S \sinh \alpha H - |T \sin \beta H| ,$$

and

$$P = (C^2 + D^2)T \sin \beta H \geq 0 \quad \text{if } T \sin \beta H \geq 0$$

$$= -(A^2 + B^2)T \sin \beta H \geq 0 \quad \text{if } T \sin \beta H \leq 0$$

Stability therefore requires $LN > M^2$ and $L, N > 0$.

Also

$$\begin{aligned} \alpha I + \beta J &= \alpha \left(I + \frac{2\beta^2}{2\alpha\beta} J \right) \\ &= \alpha \left[I + \frac{\sqrt{ac} - b}{(\lambda + \mu)(1-2\delta)\sqrt{a}} \right] \\ &= \frac{\alpha[\sqrt{ac} - \mu(\lambda + 2\mu)(1-2\delta)^2]}{(\lambda + \mu)(1-2\delta)\sqrt{a}} \end{aligned}$$

and

$$\begin{aligned} \alpha J - \beta I &= \beta \left(\frac{2\alpha^2}{2\alpha\beta} J - I \right) \\ &= \frac{\beta[\sqrt{ac} + \mu(\lambda + 2\mu)(1-2\delta)^2]}{(\lambda + \mu)(1-2\delta)\sqrt{a}} \end{aligned}$$

so that

$$U = \frac{4\mu\alpha\beta^2}{(\lambda+\mu)^2(1-2\delta)} [\sqrt{ac} + \mu(\lambda+2\mu)(1-2\delta)^2]$$

$$V = \frac{4\mu\alpha^2\beta}{(\lambda+\mu)^2(1-2\delta)} [\sqrt{ac} - \mu(\lambda+2\mu)(1-2\delta)^2]$$

Also

$$\sqrt{a} = (\lambda+2\mu)(1-e)(1+\epsilon)$$

and

$$\sqrt{\frac{c}{a}} = \frac{\sqrt{ac}}{a} = \frac{\sqrt{ac}}{(\lambda+2\mu)(1-2\delta)[(\lambda+2\mu)+2\lambda\delta]}$$

Hence

$$\begin{aligned} S &= \alpha\{\mu(1-2\delta) + \frac{\sqrt{ac}}{(\lambda+2\mu)(1-2\delta)} \\ &\quad - \frac{\lambda-\mu}{(\lambda+\mu)(\lambda+2\mu)(1-2\delta)} [\sqrt{ac} - \mu(\lambda+2\mu)(1-2\delta)^2]\} \\ &= \frac{2\mu\alpha}{(\lambda+\mu)(\lambda+2\mu)(1-2\delta)} [\sqrt{ac} + \lambda(\lambda+2\mu)(1-2\delta)^2] > 0 \end{aligned}$$

and

$$T = \frac{2\mu\beta}{(\lambda+\mu)(\lambda+2\mu)(1-2\delta)} [\sqrt{ac} - \lambda(\lambda+2\mu)(1-2\delta)^2]$$

Thus

$$\begin{aligned} U^2 + V^2 &= \frac{16\mu^2\alpha^2\beta^2}{(\lambda+\mu)^4(1-2\delta)^2} \{ (\alpha^2 + \beta^2)[ac + \mu^2(\lambda+2\mu)^2(1-2\delta)^4] \\ &\quad - 2(\alpha^2 - \beta^2)\mu(\lambda+2\mu)(1-2\delta)^2\sqrt{ac} \} \\ &= \frac{16\mu^2\alpha^2\beta^2\sqrt{ac}}{(\lambda+\mu)^2} \\ &= SU - TV . \end{aligned}$$

Now, for $H \rightarrow \infty$, we have:

$$L \rightarrow e^{\alpha H} [S + (S-U) \cos \beta H - (T+V) \sin \beta H]$$

$$\geq e^{\alpha H} [S - \sqrt{(S-U)^2 + (T+V)^2}]$$

$$N \rightarrow e^{\alpha H} [S - (S-U) \cos \beta H + (T+V) \sin \beta H]$$

$$\geq e^{\alpha H} [S - \sqrt{(S-U)^2 + (T+V)^2}]$$

$$M \rightarrow e^{\alpha H} [(T+V) \cos \beta H + (S-U) \sin \beta H]$$

$$LN - M^2 \rightarrow e^{2\alpha H} \{S^2 - [(S-U)^2 + (T+V)^2]\}$$

$$= e^{2\alpha H} [2(SU-TV) - (U^2 + V^2) - T^2]$$

$$= e^{2\alpha H} [(U^2 + V^2) - T^2] .$$

For stability, we require $LN > M^2$ and $L, N > 0$. From the preceding, the first inequality insures the second.

Hence the criterion for stability of a column of infinite depth is:

$$(U^2 + V^2) - T^2 > 0$$

or

$$4\alpha^2 (\lambda + 2\mu)^2 (1 - 2\delta)^2 \sqrt{ac} > [\sqrt{ac} - \lambda(\lambda + 2\mu)(1 - 2\delta)^2]^2$$

or

$$2(\lambda + 2\mu)^2 (1 - 2\delta)^2 [c + b\sqrt{\frac{c}{a}}] > ac - 2\lambda(\lambda + 2\mu)(1 - 2\delta)^2 \sqrt{ac} \\ + \lambda^2 (\lambda + 2\mu)^2 (1 - 2\delta)^4$$

or

$$2(\lambda + 2\mu)(1 - 2\delta)^2 \sqrt{\frac{c}{a}} [\lambda a + (\lambda + 2\mu)b] > c[a - 2(\lambda + 2\mu)^2 (1 - 2\delta)^2] \\ + \lambda^2 (\lambda + 2\mu)^2 (1 - 2\delta)^4$$

or

$$(\lambda+2\mu) \sqrt{c} [(\lambda+2\mu) - \delta(\lambda+7\mu) - 2\delta^2(3\lambda-\mu)] \\ > -\sqrt{a}[\mu(\lambda+2\mu) - \delta(\lambda+2\mu)(\lambda+7\mu) + 2\delta^2(\lambda^2 + 9\lambda\mu + 12\mu^2)]$$

The critical value is, therefore a root of:

$$(\lambda+2\mu)[(\lambda+2\mu) - \delta(\lambda+7\mu) - 2\delta^2(3\lambda-\mu)]^2[(\lambda+2\mu)^2 - 2\delta(\lambda^2 + 6\lambda\mu + 6\mu^2)] \\ = [(\lambda+2\mu) + 2\lambda\delta][\mu(\lambda+2\mu) - \delta(\lambda+2\mu)(\lambda+7\mu) + 2\delta^2(\lambda^2 + 9\lambda\mu + 12\mu^2)]^2$$

or

$$16\delta^5(5\lambda^4 + 37\lambda^3\mu + 73\lambda^2\mu^2 + 39\lambda\mu^3 + 6\mu^4) - 16\delta^4(\lambda+2\mu)(\lambda^3 - 9\lambda^2\mu - 42\lambda\mu^2 - 14\mu^3) \\ - 4\delta^3(\lambda+2\mu)(9\lambda^3 + 67\lambda^2\mu + 91\lambda\mu^2 - 31\mu^3) + 8\delta^2(\lambda+2\mu)^2(\lambda^2 - 2\lambda\mu - 17\mu^2) \\ + 4\delta(\lambda+2\mu)^2(\lambda^2 + 8\lambda\mu + 13\mu^2) - (\lambda+2\mu)^3(\lambda+3\mu) = 0$$

For $\lambda \gg \mu$, the positive roots are:

$$\delta = .285, .380, \text{ and } \frac{1}{2}$$

The larger two roots must be ruled out since, for $\delta > 3/10$, we have $b^2 - ac > 0$. It is easy to verify that the first root is the actual critical value.

For $\lambda = \mu$, the positive roots are:

$$\delta = \frac{3}{10}, .342, \text{ and } .343.$$

The larger two roots must again be discarded since, for $\delta > .316$, $b^2 - ac > 0$.

For $\lambda \ll \mu$, the positive roots are:

$$\delta = .305, .323, \text{ and } \frac{1}{3}.$$

Here only the largest root must be discarded to keep

$ac > b^2$. However, the .323 root is extraneous, introduced by the squaring during the derivation of the final equation.

The values of e are obtainable from the relation:

$$e = 1 - \sqrt{1-2\delta}$$

and, therefore, the critical strains for infinitely deep sections are:

$$\begin{aligned} e &= .344 \quad \text{for } \lambda \gg \mu \\ &= .368 \quad \text{for } \lambda = \mu \\ &= .372 \quad \text{for } \lambda \ll \mu . \end{aligned}$$

These values are considerably below the limit of $e = 1/2$ for all ratios of λ to μ obtained using the standard strain energy formula.

6. Evaluation of Buckling Strain for Small Sections.

For H and δ small, we have

$$\sqrt{ac} \rightarrow (\lambda + 2\mu)^2$$

so that

$$T > 0 ,$$

$$V > 0 ,$$

also

$$\sin \beta H > 0$$

and

$$\begin{aligned} N &= T(\cosh \alpha H - 1) \sin \beta H + S \sinh \alpha H (1 - \cos \beta H) \\ &\quad + V \cosh \alpha H \sin \beta H + U \sinh \alpha H \cos \beta H > 0 . \end{aligned}$$

The condition for stability is, therefore,

$$LN - M^2 > 0 \quad .$$

Let

$$K = \frac{\beta \sinh \alpha H - \alpha \sin \beta H}{\beta \sinh \alpha H + \alpha \sin \beta H} \quad .$$

Then

$$0 < K < 1 \quad ,$$

and the condition for stability becomes:

$$\begin{aligned} & (S \sinh \alpha H - T \sin \beta H)^2 \\ & > [(S-U)^2 + (T+V)^2] (\sinh^2 \alpha H \cos^2 \beta H + \cosh^2 \alpha H \sin^2 \beta H) \\ & = [(S-U)^2 + (T+V)^2] (\sinh^2 \alpha H + \sin^2 \beta H) \quad . \end{aligned}$$

But

$$\frac{\sinh \alpha H}{\sin \beta H} = \frac{\alpha(1+K)}{\beta(1-K)}$$

making the stability condition:

$$[\alpha S(1+K) - \beta T(1-K)]^2 > [(S-U)^2 + (T+V)^2] [\alpha^2(1+K)^2 + \beta^2(1-K)^2]$$

or

$$\begin{aligned} & \{ (\alpha S + \beta T)^2 - (\alpha^2 + \beta^2) [(S-U)^2 + (T+V)^2] \} K^2 \\ & + 2 \{ (\alpha^2 S^2 - \beta^2 T^2) - (\alpha^2 - \beta^2) [(S-U)^2 + (T+V)^2] \} K \\ & + \{ (\alpha S - \beta T)^2 - (\alpha^2 + \beta^2) [(S-U)^2 + (T+V)^2] \} > 0 \quad . \end{aligned}$$

Now

$$\begin{aligned} & (\alpha S \pm \beta T)^2 - (\alpha^2 + \beta^2) [(S-U)^2 + (T+V)^2] \\ & = (\alpha^2 + \beta^2) [2(SU - TV) - (U^2 + V^2)] - (\beta S \mp \alpha T)^2 \\ & = \frac{16\mu^2 \alpha^2 \beta^2 (\alpha^2 + \beta^2) \sqrt{ac}}{(\lambda + \mu)^2} - (\beta S \mp \alpha T)^2 \quad . \end{aligned}$$

Also

$$\beta S - \alpha T = \frac{4\lambda\mu\alpha\beta(1-2\delta)}{\lambda+\mu}$$

$$\beta S + \alpha T = \frac{4\mu\alpha\beta\sqrt{ac}}{(\lambda+\mu)(\lambda+2\mu)(1-2\delta)}$$

and

$$\begin{aligned} (\alpha^2 S^2 - \beta^2 T^2) - (\alpha^2 - \beta^2)[(S-U)^2 + (T+V)^2] \\ = (\beta S - \alpha T)(\beta S + \alpha T) + (\alpha^2 - \beta^2)[2(SU - TV) - (U^2 - V^2)] \\ = (\beta S - \alpha T)(\beta S + \alpha T) + \frac{16\mu^2 \alpha^2 \beta^2 (\alpha^2 - \beta^2) \sqrt{ac}}{(\lambda+\mu)^2} \end{aligned}$$

The stability condition may thus be written as:

$$[c - \lambda^2(1-2\delta)^2]K^2 + 2\left[\frac{b}{a} + \frac{\lambda}{\lambda+2\mu}\right]\sqrt{ac} K + \left[1 - \frac{a}{(\lambda+2\mu)^2(1-2\delta)^2}\right]c > 0,$$

or

$$\begin{aligned} \mu(\lambda+2\mu)(1-2\delta)(1-3\delta)K^2 + (\lambda+2\mu)[(\lambda+2\mu) - \delta(\lambda+7\mu) - 2\delta^2(3\lambda-\mu)]K\sqrt{\frac{c}{a}} \\ - \delta[(\lambda+2\mu)^2 - 2\delta(\lambda^2 + 6\lambda\mu + 6\mu^2)] > 0. \end{aligned}$$

Expanding K as a power series in H, we find:

$$\begin{aligned} K &= (\alpha^2 + \beta^2) \frac{H^2}{12} \left[1 - (\alpha^2 - \beta^2) \frac{H^2}{30} - (\alpha^4 + \alpha^2\beta^2 + \beta^4) \frac{11H^4}{2520} - \dots \right] \\ &= \frac{H^2}{12} \sqrt{\frac{c}{a}} \left[1 - \frac{b}{a} \frac{H^2}{30} - \left(\frac{3c}{a} + \frac{b^2}{a^2} \right) \frac{11H^4}{2520} - \dots \right] \end{aligned}$$

Now

$$\frac{b}{a} = 1 - \frac{2\delta(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2} - \frac{2\delta^2(\lambda+\mu)(\lambda^2+3\lambda\mu+4\mu^2)}{(\lambda+2\mu)^3} - \dots$$

$$\frac{c}{a} = 1 - \frac{4\delta(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2} - \frac{8\delta^2\lambda(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^3} - \dots$$

Let $H^2/12 = A_1\delta + A_2\delta^2 + A_3\delta^3 + \dots$ at the limit of stability. Then

$$K\sqrt{\frac{c}{a}} = \delta A_1 + \delta^2[A_2 - \frac{2}{5} A_1^2 - \frac{4(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2} A_1] + \dots$$

$$K^2 = \delta^2 A_1^2 + \dots$$

Substituting into the stability condition (with an equality instead of an inequality sign so as to correspond to the critical limit), and considering successive powers of δ , we find:

$$A_1 = 1$$

$$\begin{aligned} \mu(\lambda+2\mu)A_1^2 + (\lambda+2\mu)^2[A_2 - \frac{2}{5} A_1^2 - \frac{4(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2} A_1] \\ - 2(\lambda+2\mu)(\lambda+7\mu)A_1 + 2(\lambda^2+6\lambda\mu+6\mu^2) = 0, \end{aligned}$$

so that

$$A_2 = \frac{22\lambda+69\mu}{5(\lambda+2\mu)}.$$

But

$$\delta = e - \frac{e^2}{2}.$$

Hence

$$\begin{aligned} \frac{H^2}{12} &= A_1(e - \frac{e^2}{2}) + A_2(e - \frac{e^2}{2})^2 + A_3(e - \frac{e^2}{2})^3 + \dots \\ &= A_1 e + (A_2 - \frac{A_1^2}{2})e^2 + (A_3 - A_2) e^3 + \dots \end{aligned}$$

If inverted, we express e as a series in powers of $H^2/12$:

$$e = B_1 \frac{H^2}{12} + B_2 (\frac{H^2}{12})^2 + B_3 (\frac{H^2}{12})^3 + \dots$$

Then

$$1 = A_1 B_1$$

$$0 = B_1 \left(A_2 - \frac{A_1}{2} \right) + B_2 A_1^2$$

so that

$$B_1 = \frac{1}{A_1} = 1$$

in agreement with the evaluation using the standard energy formula. However

$$\begin{aligned} B_2 &= - \frac{B_1}{A_1^2} \left(A_2 - \frac{A_1}{2} \right) \\ &= - \frac{39\lambda + 128\mu}{10(\lambda + 2\mu)} \\ &= - \frac{39}{10} \quad \text{for } \lambda \gg \mu \\ &= - \frac{167}{30} \quad \text{for } \lambda = \mu \\ &= - \frac{32}{5} \quad \text{for } \lambda \ll \mu \end{aligned}$$

which are considerably more negative than the value $-7/5$ obtained from the standard energy formula. Thus, as expected, the Murnaghan strain energy density formula leads to smaller values of strain causing buckling of a column than does the standard formula, but for thin columns where buckling occurs at very low values of strain, both formulae yield the Euler value as a limit.

Footnotes

- 1 See, for example, "Theory of Elastic Stability" by S. Timoshenko, McGraw-Hill, New York, 1936.
- 2 "Determination of Buckling Criteria by Minimization of Total Energy" by Samuel Lubkin, Report IMM-NYU-241, Institute of Math. Sci., New York Univ., 1957.
- 3 As stated, for example, in "A Treatise on the Mathematical Theory of Elasticity" by A.E.H. Love, Dover, New York, 1944, p. 102
- 4 "Plane Strain Problems for a Perfectly Elastic Material of Harmonic Type" by Fritz John, Communications on Pure and Applied Mathematics, vol. XIII, No. 2, May 1960.
- 5 "Finite Deformations of an Elastic Solid" by F. D. Murnaghan, John Wiley + Sons, New York, 1951.
- 6 "The Compression of 39 Substances to 100,000 KG/CM²" by P. W. Bridgeman, Proc. American Academy of Arts and Science, vol. 76, pp. 55-70 (1948).
- 7 In that it involves, in simple form, invariants of the matrix $\eta = pp^*$ instead of $e = \sqrt{pp^*}$ (notation of Fritz John, op. cit.). However, the present paper as well as the author's attempt to use it in other applications indicates that mathematical manipulations are generally more

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13. ABSTRACT

Buckling criteria are derived for a rectangular column for large two-dimensional strains, using a strain energy density function suggested by Murnaghan. This function is analogous to the classical density function in that it involves two elastic constants but is simpler in certain respects. Despite the apparent simplicity, the evaluation proved more difficult than the equivalent analysis using the classical function.

As expected, Murnaghan's density function leads to smaller values of critical strains than does the classical but the difference is slight for slender columns. For infinitely thick columns, the critical strain is found to range between 0.344 to 0.372 depending upon the ratio of elastic constants as compared to 0.5 for the classical formula independent of this ratio.

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